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NON-LINEAR FORCED VIBRATIONS OF PLATES BY AN ASYMPTOTIC–NUMERICAL METHOD

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Non-linear forced vibrations of thin elastic plates have been investigated by an asymptotic–numerical method (ANM). Various types of harmonic excitation forces such as distributed and concentrated are considered. Using the harmonic balance method and Hamilton's principle, the equation of motion is converted into an operational formulation. Based on the finite element method a starting point corresponding to a non-linear solution associated to a given frequency and amplitude of excitation is computed. Applying perturbation techniques in the vicinity of this solution, the non-linear governing equation obtained is transformed into a sequence of linear problems having the same stiffness matrix. Employing one matrix inversion, a large number of terms of the perturbation series of the displacement and frequency can be easily computed with a small computation time. Iterations of this method lead to a powerful path-following technique. Comprehensive numerical tests for forced vibrations of plates subjected to time-harmonic lateral excitations are reported.

1. INTRODUCTION

There is a growing interest in the large-amplitude vibration problems of plates particularly in aircraft and aerospace industries. Plate elements, largely composing aerospace vehicle structures, are often forced to vibrate at large amplitudes during certain phases of flight. Consequently, non-linear vibration of plates is of considerable interest and a number of papers have been written on the subject. It is well known, for example, that when a plate is deflected more than approximately one-half of its thickness, especially if in-plane edge constraints are present, a significant geometrical non-linearity is induced [1]. This non-linearity causes an increase of the resonance frequency with the amplitude of vibration. The non-linear mode shape is amplitude dependent and changes during the period. The jump phenomenon and its corresponding multivalued region in the non linear frequency–response curves can be shown. Other phenomena involving harmonic distortion of the response and internal resonance can also be encountered.

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The dynamic behaviour of thin plates is governed by non-linear partial differential equations. Generally, explicit solutions of these equations are not available in the literature and analytical solutions exist only for few simple cases. The modal analysis and linearizing techniques are routinely used to examine the dynamic response of plate-like structures. However, standard procedures are based on the assumption of linearity and can fail to give accurate results when the amplitude of vibration is large enough to introduce significant non-linear behaviours. Therefore, it is of crucial interest for designers, for obvious safety reasons, to know how far the characteristics of real dynamic responses deviate from those defined via linear theory. Among the different numerical methods available, the finite element method is undoubtedly the most versatile one. The significant advantage of the method proposed here is the combination of the finite element method and perturbations method providing an efficient algorithm for solving the obtained non-linear problem. The applicability of this method to large-amplitude free vibrations of plates has been presented in previous works [2, 3].

A significant number of investigations have been conducted on large vibration amplitudes of plates. A summarization of the knowledge existing in the field of vibrations of plates has been presented by Leissa in his monograph [4]. The finite element modelling of laminated and anisotropic plates and shells has been discussed by Reddy in a review paper [5]. In some other works, Reddy developed and reviewed a refined shear deformation theory for composite plates [6]. A survey paper summarizing research activities on the dynamics of composite and sandwich plates during the period 1979–1981 has been presented by Bert [7]. A comprehensive set of corresponding references has been presented by Sathyamoorthy [8, 9]. The effects of transverse shear deformation, rotatory inertia, anisotropy, initial imperfections and variable rigidity on the vibration behaviour of plates are particularly reviewed and discussed in reference [9]. A review of the technical literature on the acoustic fatigue of beams and plates and some experimental investigations at higher level of dynamic response have been presented by Wolfe [10]. Recently, a review of the literature and a survey of methodological approaches of non-linear vibrations of beams and plates have been presented [11]. Attention was particularly focused on the various analytical, semi-analytical and numerical methods used in various studies related to the subject. Various works assume the dependence in time to be harmonic and combine the FEM with linearizing techniques [12–15] or with iterative–incremental procedures [16, 17]. A reduced basis technique based on FEM was developed and applied to non-linear vibrations by Noor *et al.* [18]. A finite element time-domain modal formulation was developed by Shi and Mei [19], Zhou *et al.* [20] and Shi *et al.* [21] and applied to non-linear vibrations. A hierarchical finite element method has been developed and applied to linear and non-linear vibrations of plates by Han and Petyt [22, 23]. Based on this method and continuation procedures, an intensive study of non-linear vibrations of plates and the effect of internal resonance have been investigated by Ribeiro and Petyt [24, 25]. The dynamic behaviour of plates at large vibration amplitudes was examined both theoretically and experimentally by Benamar [26] and Benamar *et al.* [27]. The second non-linear mode for various plate aspect ratios at large amplitudes has been recently analyzed by El Kadiri *et al.* [28]. Non-linear vibrations of laminated plates have been investigated by Woo and Nair [29] and by Kant and Kommineni [30] using a refined theory.

Following the previous reviews and papers, it can be observed that free vibrations of plates have been largely studied by a wide variety of researchers. Unfortunately, forced vibrations have not attracted so many workers and only few papers containing numerical results can be found. Using the single-mode approach and elliptic functions, the non-linear forced oscillations have been analyzed by Hsu [31] where elliptic and harmonic excitations are taken into account. Based on Hamilton's principle and perturbation procedures, large-amplitude forced vibrations of beams and plates have been analytically presented by Rehfield [32]. Free and steady state vibrations of plates using FEM have been studied by Wellford *et al.* [33]. Intensive and instructive studies of forced vibrations of structures based on the finite element method have been presented by Mei *et al.* [13, 14], Chiang *et al.* [15] Shi *et al.* [19,21] and Zhou *et al.* [20]. Based on Galerkin's method and the harmonic-balance method applied to a single-degree-of-freedom system, some frequency–amplitude relationships were presented by Dumir and Bhaskar [34] for beams and plates and by Sherif [35] for unsymmetric sandwich circular plates. A direct numerical integration method applied to the resulting Duffing equation has been used by Singh *et al.* [36] to analyze the forced vibration of antisymmetric rectangular cross-ply plates. The Fourier series has been used by Teng *et al.* [37] to study the non-linear forced vibration of rectangular plates. A semi-analytical method, based on the multi-mode analysis and continuation procedures, has been developed and applied to forced vibrations of beams at large amplitudes [38, 39].

The purpose of the present paper is to take advantage of coupling the FEM and perturbation methods in order to study the non-linear forced vibrations of plates. The effectiveness of this coupling has been demonstrated using ANM. It has been successfully used for computing perturbed bifurcation branches of beams by Damil and Potier-Ferry [40], for post-buckling behaviours of plates and shells by Azrar *et al.* [41] and extended to some elastostatic problems by Cochelin [42]. This method has been applied by various colleagues to different non-linear problems; see for instance Zahrouni *et al.* [43], Elhage-Hussein *et al.* [44] and Daya and Potier-Ferry [45]. Its applicability to non-linear free vibrations of plates has been presented in previous works [2, 3]. In this paper, various types of harmonic excitations such as uniformly distributed and concentrated forces are considered. The use of the harmonic-balance method permits one to obtain operational formulations. Based on the Newton–Raphson algorithm, a starting point corresponding to given frequency and amplitude of excitation is obtained. ANM is applied for the solution in the vicinity of this starting point and a large part of the non-linear solution path is obtained. The non-linear resonance curves are investigated by iterating the ANM. Comprehensive numerical tests for forced vibrations of plates with various boundary conditions are reported.

2. REVIEW OF MATHEMATICAL FORMULATIONS

2.1. BASIC FUNCTIONAL

Let the displacement components of the middle surface of the plate be u, v and w, where u and v are the in-plane displacements and w the transverse displacement in the x, y and z directions. Assuming that the plate is thin, the non-linear strain–displacement relationships associated with von Karman plate theory are given by

$$\mathbf{\Gamma}^L = \left\{ \begin{array}{c} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{array} \right\}, \quad \mathbf{\Gamma}^{NL} = \left\{ \begin{array}{c} \frac{1}{2}(\partial w/\partial x)^2 \\ \frac{1}{2}(\partial w/\partial y)^2 \\ (\partial w/\partial x)/(\partial w \partial y) \end{array} \right\}, \quad \boldsymbol{\kappa} = \left\{ \begin{array}{c} -\partial^2 w/\partial x^2 \\ -\partial^2 w/\partial y^2 \\ -2\partial^2 w/\partial x \partial y \end{array} \right\}, \quad (1)$$

where $\mathbf{\Gamma} = \mathbf{\Gamma}^L + \mathbf{\Gamma}^{NL}$ is the generalized membrane strain and $\boldsymbol{\kappa}$ the bending strain. The bending strain is assumed to be linear with respect to the displacement (framework with moderate rotations). The in-plane forces **N** and bending moments **M** are assumed to be related to the strain and curvature by the constitutive relations

$$\mathbf{N} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [\mathbf{C}_m] : \boldsymbol{\Gamma},$$

$$\mathbf{M} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [\mathbf{C}_b] : \boldsymbol{\kappa}. \tag{2}$$

in which $[\mathbf{C}_m]$ and $[\mathbf{C}_b]$ are the symmetric stiffness matrices. The more general situation with the coupling matrix $[\mathbf{C}_{mb}]$ between membrane and flexure, which is of great concern for antisymmetric composite laminates, will not be considered here for simplicity. The elastic strain energy V of plates is given by

$$V = \frac{1}{2}\int_{\Omega} (\boldsymbol{\Gamma} : [\mathbf{C}_m] : \boldsymbol{\Gamma} + \boldsymbol{\kappa}[\mathbf{C}_b] : \boldsymbol{\kappa})\, \mathrm{d}\Omega, \tag{3}$$

where Ω is the middle surface of the plate. Since $\boldsymbol{\Gamma}$ is quadratic in w, the functional V is of degree 4 with respect to w. To reduce the degree of the non-linearity, the mixed Hellinger–Reissner functional is used [2, 3]:

$$\mathbf{H}(u, v, w, \mathbf{N}) = \int_{\Omega} (\mathbf{N} : \boldsymbol{\Gamma} - \tfrac{1}{2}\mathbf{N} : [\mathbf{C}_m]^{-1} : \mathbf{N} + \tfrac{1}{2}\boldsymbol{\kappa} : [\mathbf{C}_b] : \boldsymbol{\kappa})\, \mathrm{d}\Omega, \tag{4}$$

where the unknown is the mixed (displacement–stress) $\mathbf{U} =^t \{u, v, w, \mathbf{N}\}$. Neglecting the rotatory inertia terms, the kinetic energy is given by

$$T = \frac{1}{2}\int_{\Omega} \rho\, h\, (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\, \mathrm{d}\Omega, \tag{5}$$

in which the dot means the differentiation with respect to time, ρ the mass density and h the thickness of the plate. Assuming that the plate is excited laterally by the force $\mathbf{F}(x, y, t)$, the associated virtual work is given by

$$W_{ext} = \int_{\Omega} \mathbf{F}(x, y, t)\, \mathbf{w}(x, y, t)\, \mathrm{d}\Omega. \tag{6}$$

2.2. HARMONIC-BALANCE METHOD AND OPERATIONAL FORMULATIONS

Because of the fundamental nature of the harmonic excitation and because it has many practical and theoretical applications, it will be the subject of this paper. One assumes that the plate is excited by a force in the form

$$\mathbf{F}(x, y, t) = \mathbf{f}(x, y)\cos(\omega t). \tag{7}$$

As presented in reference [3] for the free vibration case, the displacement vector is assumed to be in the following form:

$$\mathbf{u}(x, y, t) = \mathbf{u}(x, y)\sin^2\omega t, \quad \mathbf{v}(x, y, t) = \mathbf{v}(x, y)\sin^2 \omega t, \quad \mathbf{w}(x, y, t) = \mathbf{w}(x, y)\sin \omega t. \tag{8}$$

in which ω is the circular frequency parameter. So, the principle of the harmonic balance method is to consider the amplitude $\mathbf{u}(x, y)$, $\mathbf{v}(x, y)$ and $\mathbf{w}(x, y)$ as the main unknowns. They will be solutions of sort of static system but accounting for inertial terms. For simplicity, these amplitudes will be designed by the same symbols as original unknowns.

The insertion of equation (8) into equations (1) and (2) gives the time dependence of the strain, curvature, and membrane tensors as follows:

$$\boldsymbol{\Gamma}^L(x,y,t) = \boldsymbol{\gamma}^L(x,y)\sin^2\omega t, \quad \boldsymbol{\Gamma}^{NL}(x,y,t) = \boldsymbol{\gamma}^{NL}(x,y)\sin^2\omega t,$$
$$\boldsymbol{\kappa}(x,y,t) = \boldsymbol{\kappa}(x,y)\sin\omega t, \quad \mathbf{N}(x,y,t) = \mathbf{N}(x,y)\sin^2\omega t. \tag{9}$$

A more general time dependence of the displacement vector for transverse and in-plane excitation is theoretically presented in a previous work [2]. Using Galerkin's method, the non-linear differential equation of Duffing type was obtained for transverse excitation and an equation of Mathieu type was obtained if the in-plane excitation is also considered.

To study the history of the solution corresponding to a period, the initial time is set as $t_0 = 0$ and the final time $t_1 = 2\pi/\omega$.

$$\int_0^{2\pi/\omega} (H - T - W_{ext})\,\mathrm{d}t = \frac{\pi}{\omega}\left\{\frac{3}{4}\int_\Omega \left(\mathbf{N}:\boldsymbol{\gamma} - \frac{1}{2}\mathbf{N}:[\mathbf{C}_m]^{-1}:\mathbf{N}\right)\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \boldsymbol{\kappa}:[\mathbf{C}_b]:\boldsymbol{\kappa}\,\mathrm{d}\Omega\right.$$
$$\left. -\omega^2\frac{1}{2}\rho h\int_\Omega (u^2+v^2+w^2)\,\mathrm{d}\Omega - \int_\Omega \mathbf{f}(x,y)\mathbf{w}(x,y)\,\mathrm{d}\Omega\right\}. \tag{10}$$

Using Hamilton's principle, one gets the governing equation becomes

$$\frac{3}{4}\int_\Omega (\delta\boldsymbol{\gamma}:\mathbf{N} + \delta\mathbf{N}:\boldsymbol{\gamma} - \delta\mathbf{N}:[\mathbf{C}_m]^{-1}:\mathbf{N})\,\mathrm{d}\Omega + \int_\Omega \delta\boldsymbol{\kappa}:[\mathbf{C}_b]:\boldsymbol{\kappa}\,\mathrm{d}\Omega$$
$$-\omega^2\rho h\int_\Omega (u\,\delta u + v\,\delta v + w\,\delta w)\,\mathrm{d}\Omega - \int_\Omega \mathbf{f}(x,y)\,\delta\mathbf{w}(x,y)\,d\Omega = 0. \tag{11}$$

This mixed variational principle can be used directly in conjunction with a mixed finite element method. To obtain a displacement formulation, one can determine the membrane stress **N** as a function of the displacement.

$$\mathbf{N}(x,y) = [\mathbf{C}_m]\{\boldsymbol{\gamma}(u,v,w)\} = [\mathbf{C}_m]\{\boldsymbol{\gamma}^L(u,v) + \boldsymbol{\gamma}^{NL}(\mathbf{w},\mathbf{w})\}. \tag{12}$$

The insertion of the stress N in equation (11) leads to a variational principle of cubic non-linearity in displacement. After discretization by the finite element method, one gets a non-linear matrix problem. This problem could be treated by a predictor–corrector method. The purpose here is to solve the variational equation (11) using an asymptotic–numerical method.

In view of using an operational notation as presented in references [2, 3], the governing equation (11) can be written as

$$\langle \mathbf{LU}, \delta\mathbf{U}\rangle - \omega^2\langle \mathbf{MU}, \delta\mathbf{U}\rangle + \langle \mathbf{Q}(\mathbf{U},\mathbf{U}), \delta\mathbf{U}\rangle = \langle \mathbf{F}, \delta\mathbf{U}\rangle, \tag{13a}$$

in which $\mathbf{U} = {}^t[u, v, w, \mathbf{N}]$ is the mixed vector and

$$\langle \mathbf{LU}, \delta\mathbf{U}\rangle = \frac{3}{4}\int_\Omega [\mathbf{N}:\delta\boldsymbol{\gamma}^L + (\boldsymbol{\gamma}^L - [\mathbf{C}_m]^{-1}:\mathbf{N}):\delta\mathbf{N}]\,\mathrm{d}\Omega + \int_\Omega \delta\boldsymbol{\kappa}:[\mathbf{C}_b]:\kappa\,\mathrm{d}\Omega, \tag{13b}$$

$$\langle \mathbf{MU}, \delta\mathbf{U}\rangle = \rho h\int_\Omega [u\,\delta u + v\,\delta v + w\,\delta w]\,\mathrm{d}\Omega, \tag{13c}$$

$$\langle \mathbf{Q}(\mathbf{U},\mathbf{U}), \delta\mathbf{U}\rangle = \frac{3}{4}\int_\Omega [\mathbf{N}:\delta\boldsymbol{\gamma}^{NL} + \boldsymbol{\gamma}^{NL}:\delta\mathbf{N}]\,\mathrm{d}\Omega, \tag{13d}$$

$$\langle \mathbf{F}, \delta\mathbf{U}\rangle = \int_\Omega \mathbf{f}(x,y)\,\delta\mathbf{w}(x,y)\,\mathrm{d}\Omega. \tag{13e}$$

The operators **L** and **M** are linear and **Q** is a quadratic one. The matrices corresponding to the operators **L** and **M** are the linear stiffness and the mass matrix respectively. The formulation $\langle \mathbf{F}, \delta\mathbf{U}\rangle$ represents the virtual work of the harmonic force.

2.3. DEFINITION OF THE APPLIED FORCE

Consider the plate excited by a distributed harmonic uniform force, $\mathbf{f}(x, y) = F^d$, and by a concentrated harmonic force applied at the point (x_0, y_0), $\mathbf{f}(x, y) = F^c\mathbf{\Delta}(x - x_0,\ y - y_0)$, in which $\mathbf{\Delta}$ is the Dirac function and F^c and F^d are constants. For these types of excitations, the formulation (13e) is then given, respectively, by

$$\langle \mathbf{F}, \delta\mathbf{U}\rangle = F^d \int_\Omega \delta\mathbf{w}(x, y)\,\mathrm{d}\Omega,\ \langle \mathbf{F}, \delta\mathbf{U}\rangle = F^c \int_\Omega \mathbf{\Delta}(x - x_0,\ y - y_0)\delta\mathbf{w}(x, y)\,\mathrm{d}\Omega. \qquad (14\text{a, b})$$

In order to use the same amplitude of excitation as in reference [14], the Galerkin-type process can be used. In the vicinity of the first resonance, the unknown **U** is assumed to be proportional to the first natural vibration mode U_{L1}:

$$\mathbf{U} = AU_{L1} \quad \text{and} \quad \delta\mathbf{U} = U_{L1}, \qquad (15)$$

where A is the amplitude parameter. The insertion of equation (15) into equation (13a) leads to the following non-linear frequency–amplitude relationship:

$$\omega^2 = \omega_{L1}^2 + \langle \mathbf{Q}(U_{L1}, U_{L1});\ U_{L1}\rangle / \langle \mathbf{M}U_{L1};\ U_{L1}\rangle A^2 - P_0/A, \qquad (16\text{a})$$

$$\omega_{L1}^2 = \langle \mathbf{L}U_{L1};\ U_{L1}\rangle / \langle \mathbf{M}U_{L1};\ U_{L1}\rangle, \qquad (16\text{b})$$

where ω_{L1} is the linear frequency associated with the linear mode U_{L1} and P_0 is the dimensionless load factor. Following formulations (14a) and (14b), P_0 is given for a harmonic distributed force and for a concentrated harmonic force respectively by

$$P_0^d = F^d \int_\Omega w_1(x, y)\,\mathrm{d}\Omega / \omega_{L1}^2 \langle \mathbf{M}U_{L1};\ U_{L1}\rangle, \qquad (17\text{a})$$

$$P_0^c = F^c w_1(x_0, y_0)/\omega_{L1}^2 \langle \mathbf{M}U_{L1}; U_{L1}\rangle \qquad (17\text{b})$$

in which $w_1(x, y)$ is the transverse displacement component of the linear mode. These formulations permitted determination of the relationships between P_0 and F^c and F^d. That will be used later to investigate the numerical results.

2.4. ASYMPTOTIC EXPANSIONS

The aim here is to use the ANM to study the non-linear forced vibrations of plates. The unknowns are the mixed vector **U** and the non-linear frequency parameter ω. One assumes that $(\mathbf{U}_0, \omega_0)$ is a solution of the non-linear equation (13a) and that, in the vicinity of this point, the solution can be represented by power series with respect to a path parameter a

$$\mathbf{U} = \mathbf{U}_0 + a\mathbf{U}_1 + a^2\mathbf{U}_2 + a^3\mathbf{U}_3 + \cdots + a^p\,\mathbf{U}_p + \cdots,$$

$$\omega^2 = \omega_0^2 + a\omega_1 + a^2\omega_2 + a^3\omega_3 + \cdots + a^p\omega_p + \cdots. \qquad (18)$$

in which U_p and ω_p are the new unknowns which have to be computed. Introducing equation (18) into equation (13a) and equating like powers of a, one obtains the set of linear problems

order 1:

$$\mathbf{L}_t\mathbf{U}_1 = \omega_1\,\mathbf{M}\mathbf{U}_0, \qquad (19\text{a})$$

order 2:

$$\mathbf{L}_t\mathbf{U}_2 = \omega_2\,\mathbf{M}\mathbf{U}_0 + \omega_1\,\mathbf{M}\mathbf{U}_1 - \mathbf{Q}(\mathbf{U}_1, \mathbf{U}_1) \qquad (19b)$$

$$\vdots$$

order p:

$$\mathbf{L}_t\mathbf{U}_p = \omega_p\,\mathbf{M}\mathbf{U}_0 + \sum_{r=0}^{p-2}\omega_{(r+1)}\,\mathbf{M}\mathbf{U}_{p-r-1} - \sum_{r=1}^{p-1}\mathbf{Q}(\mathbf{U}_r,\,\mathbf{U}_{p-r}) \qquad (19c)$$

in which the tangent operator $\mathbf{L}_t$ is defined as $\mathbf{L}_t(\cdot) = \mathbf{L}(\cdot) + 2\,\mathbf{Q}(\mathbf{U}_0, \cdot)$. The first equation corresponds to the linearization of equation (13a) at the starting point $(\mathbf{U}_0, \omega_0)$, i.e., the vector U_1 and the coefficient ω_1 correspond to the tangent of the branch at the starting point. Notice that at each order p, both $\mathbf{U}_p$ and ω_p are unknowns and there is one superfluous unknown in each of these linear problems. So, a solvability equation including the displacement vector and the frequency parameter following the idea of the arc length measure is added [42]. The solution procedure followed in this study is presented in Appendix A.

2.5. NUMERICAL SOLUTION PROCEDURES

Based on the finite element method (FEM), the non-linear forced vibration of square and rectangular plates with various boundary conditions are analyzed by this method. The triangular shell elements DKT, which have three nodes and five d.o.f. per node (u, v, w, θ_x, θ_y) are used for the discretization of the plate. Due to the symmetry, only a quarter of plate is modelled for symmetric modes and a half for the antisymmetric ones.

2.5.1. *Linear forced vibration*

Before computing the non-linear response curves given by a numerical solution of equation (13a), it is more convenient to start by the linear solution. The linear forced vibration of plate is modelled by the following equation obtained by neglecting the non-linear term in equation (13a):

$$\langle \mathbf{L}\mathbf{U}_L,\,\delta\mathbf{U}\rangle - \omega_L^2\langle \mathbf{M}\mathbf{U}_L,\,\delta\mathbf{U}\rangle = \langle \mathbf{F},\,\delta\mathbf{U}\rangle. \qquad (20)$$

After the finite element discretization, one obtains the following matrix problem:

$$[\mathbf{Ke}]\{\bar{\mathbf{U}}_L\} - \omega_L^2[\mathbf{M}]\{\bar{\mathbf{U}}_L\} = \{\mathbf{F}\}, \qquad (21)$$

in which $[\mathbf{Ke}]$ is the elastic stiffness matrix, $[\mathbf{M}]$ is the mass matrix, $\{\bar{\mathbf{U}}_L\}$ is the displacement vector at nodes and $\{\mathbf{F}\}$ is the force vector. This problem has been numerically solved by using an incremental procedure of the frequency and solving the associated linear problems. Numerical solution of equation (21) gives linear responses and the associated linear frequencies corresponding to excitation $\mathbf{F}$. The linear resonance curves can be easily obtained for various amplitudes of excitation. Neglecting the excitation force, the linear free vibration of plates is modelled by the eigenvalue problem

$$[\mathbf{Ke}]\{\bar{\mathbf{U}}_1\} = \omega_L^2[\mathbf{M}]\{\bar{\mathbf{U}}_1\}. \qquad (22)$$

Numerical solution of this problem gives linear modes and associated natural frequencies of vibration of plates.

2.5.2. *Non-linear forced vibration*

The non-linear free vibration of plates has been largely studied by this method and presented in references [2,3]. The associated non-linear frequency–displacement curves are the backbone curves that bifurcate from the fundamental solution ($\mathbf{U} = 0$) at eigenfrequencies given by equation (22). The asymptotic developments are done in the vicinity of the bifurcating point and a succession of linear problems are numerically solved [2, 3]. While free vibrations lead to bifurcation phenomena, forced vibrations do not. An exception is the appearance of subharmonics or superharmonics phenomena, which will not be treated in this study. As there are no bifurcation points in the kind of forced vibrations that will be studied, a starting point is needed in order to use the perturbation procedure in its vicinity. This starting point, ($\mathbf{U}_0$, ω_0), is the solution of the non-linear problem

$$\mathbf{L}\mathbf{U}_0 - \omega_0^2\,\mathbf{M}\mathbf{U}_0 + \mathbf{Q}(\mathbf{U}_0,\,\mathbf{U}_0) = F. \tag{23}$$

In this study, this problem has been numerically solved by the Newton–Raphson iterative procedure for a fixed excitation amplitude F and for a given frequency ω_0. This allows the solution $\mathbf{U}_0$ corresponding to a given ω_0 and F. This known solution is used as a starting point for the asymptotic numerical method. The unknowns U_p and ω_p in the polynomial representation (18) of the non-linear solution ($\mathbf{U}$, ω) are given by a recurrent solution of the linear problems (19) with only one matrix inversion per step. The procedure for solution at the non-linear responses is presented in Appendix A.

The polynomial solutions (18) coincide almost perfectly inside the radius of convergence but they diverge out of this zone of validity [2, 3, 40–42]. This limit can be computed automatically using the following criterion [42]:

$$a_{series} = (\varepsilon\|\mathbf{U}_1\|/\|\mathbf{U}_n\|)^{1/(n-1)}, \tag{24}$$

where ε is a small given number.

Note that this simple criterion gives a good order of magnitude of the area of validity of the solution, whilst requiring almost no computing time. Taking a starting point in the zone of validity of the solution, one can reapply the ANM and go far in the solution path [3, 42]. Although, the continued solution has a radius of convergence, the application of the ANM iteratively allows one to determine a complex non-linear branch.

For the sake of clarity, the path-following procedure used in this study is presented in Figure 1. For free vibrations, the starting point b1 ($\mathbf{U} = 0$, $\omega = \omega_{\mathrm{L}}$) corresponding to linear free vibrations is given by a numerical solution of the eigenvalue problem (22). Applying the ANM in the neighbourhood of this point permitted the backbone curve to be obtained up to the radius of convergence [2]. This limit b2 is easily computed by the criterion (24). Taking b2 as a starting point and reapplying the ANM permitted, a further part of the solution to be calculated, also limited by a radius of convergence b3. This procedure has been followed for non-linear free vibration of plates and presented in paper [3].

For forced vibration, the starting point f1 corresponding to a given frequency ω_0 (here $\omega_0 = 0$) is obtained by a numerical solution of the non-linear problem (23) using the Newton–Raphson algorithm. Applying the ANM in the vicinity of f1, part of the solution (f1–f2) is obtained in one step. Applying iteratively the ANM permitted one to calculate the non-linear solution (f1 - f2 - f3 - $\cdots$) up to a desired amplitude. It is well known that, at the frequency corresponding to the solution f3, the problem has three solutions denoted as f3, f4 and f6. The polynomial solution given by the ANM remained in the same solution branch. In order to jump to another branch, a starting point in the desired branch is

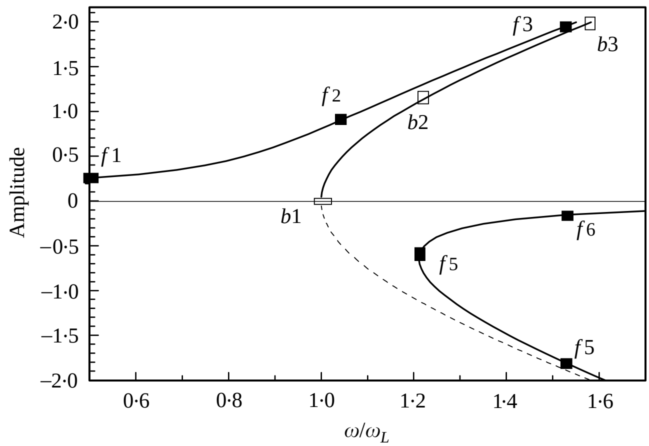

Figure 1. Forced vibrations of plates by the asymptotic numerical method. Path-following techniques used in the analysis. b1, b2 and b3 are the starting points for free vibrations. f_p are starting points for forced vibrations.

needed. This starting point is computed by the Newton–Raphson algorithm with judicious initial conditions. The solution corresponding to f4 is computed using opposite displacements corresponding to f3 as initial conditions. f6 can be computed by taking $(u = v = w = 0)$ as the initial conditions. The application of the ANM iteratively allows one to determine non-linear response functions of forced vibrations by a succession of local asymptotic expansions.

3. RESULTS AND DISCUSSION

Based on the ANM, numerical solutions were performed for non-linear forced vibrations of square and rectangular isotropic plates with length L and width l. The plate is modelled with triangular shell elements DKT which have three nodes and five d.o.f. per node $(u, v, w, \theta_x, \theta_y)$. For symmetry reasons, only a quarter of the plate has been discretized with 121 nodes. The boundary conditions concerned in the present study are simply supported $\{(u = v = w = \theta_x = 0$ at $x = 0$ and $x = L)$ and $(u = v = w = \theta_y = 0$ at $y = 0$ and $y = l)\}$ and fully clamped $(u = v = w = \theta_x = \theta_y = 0)$ at all edges. The small parameter ε in equation (24) is set at $\varepsilon = 10^{-4}$. Non-linear frequency–response curves associated with harmonic uniform distributed and concentrated forces are computed. In order to make comparisons between numerical results available based on the finite element method or on analytic procedures, the same formulation of the excitation forces as that presented by Mei *et al.* [14, 15] is considered, as explained in section 2.3. Frequency ratios for non-linear vibration of fully clamped and simply supported isotropic square plates under a harmonic uniformly distributed force are presented in Tables 1 and 2. The case of a concentrated harmonic force at the centre of simply supported square plates is presented in Table 3. It may be noticed from these tables that results obtained by the presented method match very well with those obtained by hierarchical finite element method and continuation procedure [24]. The non-linear problem was solved by both methods without any simplification. In comparison with single-mode analysis using elliptic or perturbation solutions, it can be shown that these predictions are good enough for large amplitudes [14, 31]. For very small amplitudes, the effect of the second mode affects the solution and some discrepancies are

TABLE 1

Frequency ratio ω/ω_L of the non-linear forced vibration of fully clamped square plates under a harmonic uniform distributed force ($L/h = 240$, $P_0^2 = 0{\cdot}2$)

Present results (ANM)		HFEM [24]†		W_{max}/h	FEM + Lin. [14]	Elliptic [14, 31]	Perturbation [14, 31]
W_{max}/h	ω/ω_L	W_{max}/h	ω/ω_L				
0·2000665	0·21599796	0·2000	0·2432	0·2	0·1033	0·1200	0·1227
200036	1·4329863	−0·2072	1·4275	−0·2	1·4183	1·4195	1·4195
0·4003584	0·75314512	—	—	0·4	0·7372	0·7483	0·7484
−0·400076	1·2505342	—	—	−0·4	1·2426	1·2490	1·2491
−0·6001276	0·89486919	0·6008	0·8971	0·6	0·8746	0·8951	0·8956
−0·600051	1·2092921	−0·59011	1·2120	−0·6	1·1966	1·2117	1·2119
0·8001071	0·99116717	—	—	0·8	0·9617	0·9941	0·9954
−0·800101	1·2148796	—	—	−0·8	1·1938	1·2203	1·2210
1·00009	1·0768604	1·0013	1·0803	1	1·0362	1·0822	1·0845
−1·00005	1·2457381	−0·9952	1·2475	−1	1·2140	1·2540	1·2555

†Hierarchical Finite Element Method and continuation procedure.

TABLE 2

Frequency ratio ω/ω_L of the non-linear forced vibration of simply supported square plates under a harmonic uniform distributed force ($L/h = 240$, $P_0^d = 0{\cdot}2$)

W_{max}/h	Present result (ANM)	W_{max}/h	Elliptic [14, 31]	Perturbation [14, 31]	FEM + Lin. [14]
0·20010024	0·23742884	0·2	0·1944	0·1987	0·1643
−0·20009037	1·4333909	−0·02	1·4281	1·4281	1·4238
0·40032886	0·81547934	0·4	0·8102	0·8111	0·7800
−0·40004806	1·2880868	−0·4	1·2874	1·2876	1·2682
0·60045967	1·0145738	0·6	1·0084	1·0110	0·9544
−0·60003548	1·2990543	−0·6	1·2983	1·2995	1·2560
0·80038172	1·1800401	0·8	1·1703	1·1755	1·0886
−0·80015950	1·3719442	−0·8	1·3686	1·3718	1·2981
1·0005105	1·3436439	1	1·3283	1·3369	1·2171
−1·0001737	1·4809742	−1	1·4726	1·4789	1·3717

shown. The difference is more pronounced with numerical results obtained by FE and linearizing procedure [14]. Non-linear frequency response curves in the vicinity of the first mode of a fully clamped square plate under concentrated harmonic forces at the centre ($P_0^c = 0{\cdot}1, P_0^c = 0{\cdot}2$) are shown in Figure 2. The backbone curve for free vibration is also plotted in this figure. Continuation steps of the ANM are presented in order to show the path-following procedure necessary for the analysis. It can be shown that several steps are needed around the limit point or the bifurcation point, but out of this zone the length of the steps increases and a large part of the solution is obtained by few steps. Linear and non-linear frequency response curves of a fully clamped square plate under harmonic distributed forces ($P_0^d = 0{\cdot}1$, $P_0^d = 0{\cdot}2$) are plotted in Figure 3. It can be seen clearly, as it was well known, that the non-linear analysis is needed for large amplitudes. The frequency

TABLE 3

Frequency ratio ω/ω_L of simply supported square plates under a concentrated harmonic force at the centre ($L/h = 240$, $P_0^c = 0{\cdot}025\ \pi^2$)

W_{max}/h	Present result (MAN)	W_{max}/h	Elliptic [14, 31]	Perturbation [14, 31]	FEM + Lin. [14]
−0·20006836	1·4572162	−0·2	1·5078	1·5077	1·4957
0·40000090	0·69986917	0·4	0·7342	0·7356	0·7129
−0·40001674	1·3385672	−0·4	1·3320	1·3322	1·3093
0·60017610	0·97157014	0·6	0·9688	0·9717	0·9212
−0·60004274	1·3624560	−0·6	1·3280	1·3291	1·2849
0·80006483	1·1691112	0·8	1·1449	1·1504	1·0698
−0·80013709	1·4475898	−0·8	1·3898	1·3929	1·3209
1·0002160	1·3544835	1	1·3103	1·3193	1·2068
−1·0000771	1·5676149	−1	1·4885	1·4946	1·3910

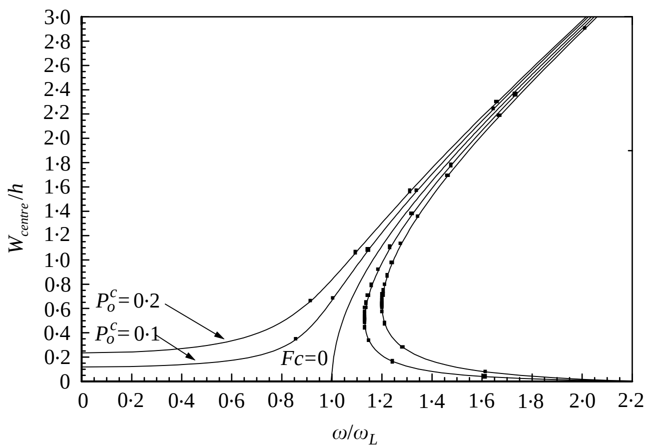

Figure 2. Non-linear frequency response function curves in the vicinity of the first mode of a fully clamped square plate under concentrated harmonic forces at the centre.

ratios ω/ω_L for fully clamped and simply supported square plates under concentrated harmonic forces at the centre are presented in Figure 4. The response functions for simply supported square and rectangular plates under a distributed harmonic force and the associated backbone curves are given in Figure 5. It appears clearly that the simply supported boundary conditions yield a larger non-linear response than the clamped ones. The non-linearity associated with rectangular plates is more pronounced than that associated with square plates. The same results were presented and discussed by Mei and co-workers using FEM and the iterative process for isotropic and composite plates [14, 15].

Non-linear deflection shapes of a fully clamped square plate under a harmonic distributed force ($P_0^d = 0{\cdot}2$) corresponding to various frequencies are presented in Figure 6. Figure 7 is added in order to show clearly the correspondence between the deflection shapes in Figure 6 and the frequency. Frequencies associated with points on

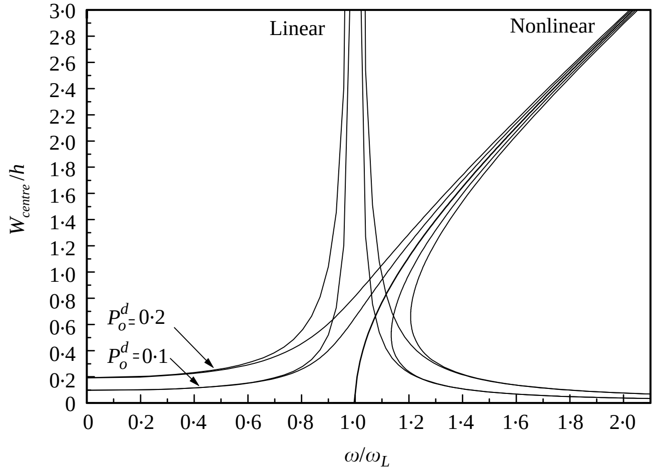

Figure 3. Linear and non-linear frequency–response function curves in the vicinity of the first mode of a fully clamped square plate under harmonic distributed forces.

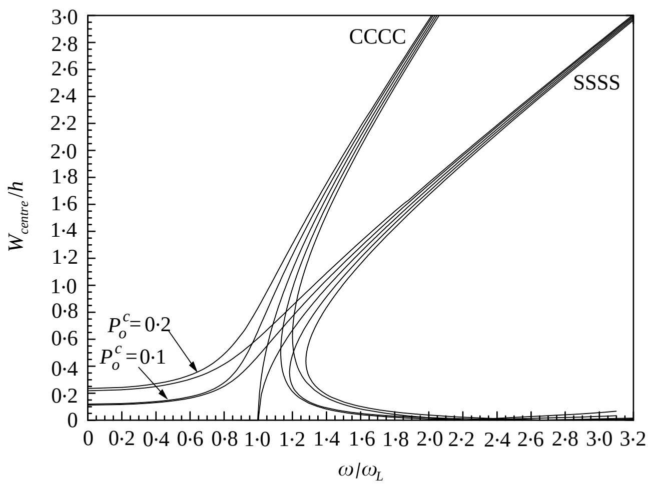

Figure 4. Forced vibrations of a fully clamped and a simply supported square plates under concentrated harmonic forces at the centre.

Figure 7 indicated by (a,b,c, 1,2,3) correspond to deflection shapes (a,b,c, 1,2,3) in Figure 6. Considering a decreasing frequency procedure, the jumping effect is produced from the limit point c to a new solution 1. The evolution of the shape deflection of the plate can be seen clearly in Figure 6. Frequency ratios for non-linear vibrations, corresponding to the first resonance, of a fully clamped rectangular plate ($L/l = 2$) under a concentrated force at the centre ($P_0^c = 0{\cdot}1$) are presented in Table 4. In Figure 8, the frequency–response curves are presented, corresponding to large frequency ranges involving the first and the third resonance of a fully clamped rectangular plate under a harmonic concentrated force. One can see the connection between the first and the third resonance. These curves are obtained using an iterative process of the ANM. In this way, any resonance curve can be automatically computed at any desired range of amplitudes.

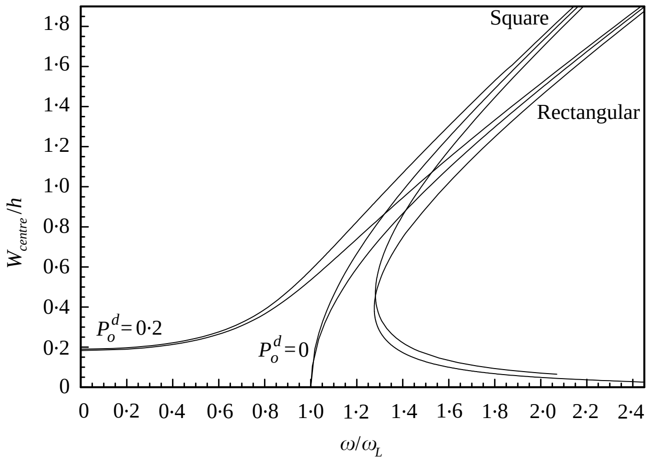

Figure 5. Free and forced vibrations of simply supported square and rectangular plates ($L/l = 2$) under a harmonic distributed force $P_0^d = 0.2$.

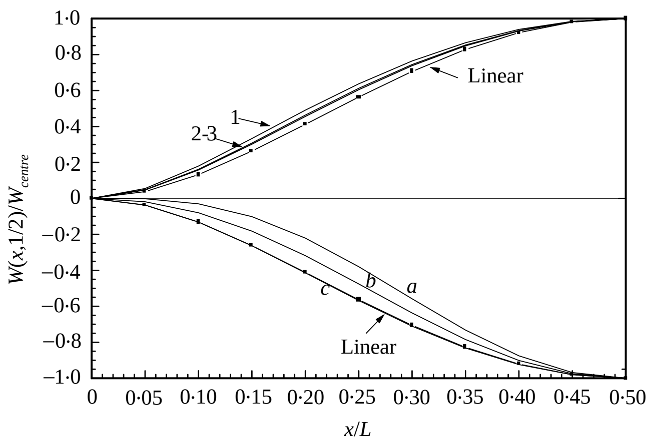

Figure 6. Non-linear forced vibration deflection shapes of a fully clamped square plate under a harmonic distributed force ($P_0^d = 1$) corresponding to decreasing frequencies. (a) $\omega/\omega_L = 2{\cdot}5$, $W_{centre}/h = -0{\cdot}25361104$; (b) $\omega/\omega_L = 2$, $W_{centre}/h = -0{\cdot}38451844$; (c) $\omega/\omega_L = 1{\cdot}5301950$; $W_{centre}/h = -1{\cdot}1239271$; (1) $\omega/\omega_L = 1{\cdot}5292688$, $W_{centre}/h = 2{\cdot}2324208$, (2) $\omega/\omega_L = 1$, $W_{centre}/h = 1{\cdot}389879$, (3) $\omega/\omega_L = 0.5$, $W_{centre}/h = 0{\cdot}9127738$.

4. CONCLUSION

An asymptotic–numerical method (ANM) to solve non-linear forced vibration of plates submitted to time-harmonic lateral excitations has been developed. Based on Von Karman plate theory, the harmonic balance method and Hamilton's principle, the dynamic problem of plate vibration was transformed into a static one. Using the finite element method and the Newton–Raphson algorithm, a starting point corresponding to the solution associated with a given frequency and amplitude of excitation was computed. The ANM was applied in the vicinity of this solution and a large part of the non-linear solution was obtained by solving a sequence of linear problems having the same stiffness matrix. Iteration of this method, leading to a path-following technique, permitted one to obtain

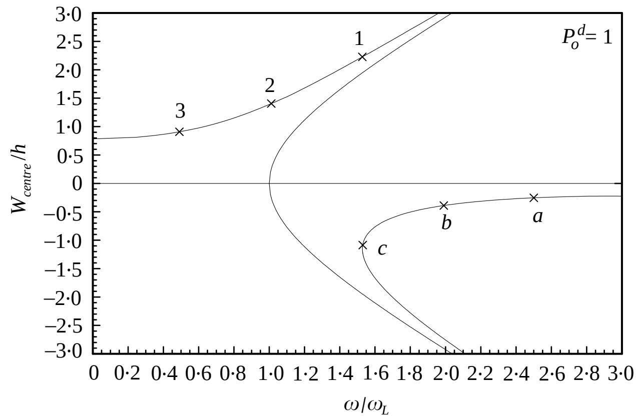

Figure 7. Forced vibrations of a fully clamped square plates under a harmonic distributed force $P_0^d = 1$.

TABLE 4

Frequency ratio ω/ω_L of a fully clamped rectangular plate under a concentrated harmonic force at the centre ($L/l = 2$, $P_0^c = 0{\cdot}1$)

W_{centre}/h	ω/ω_L	W_{centre}/h	ω/ω_L
−0·2000476	1·179367	0·2001011	0·5682657
−0·4000036	1·132928	0·4002796	0·8726742
−0·600037	1·139342	0·6000040	0·9694668
−0·800116	1·171764	0·8003387	1·044560
−1·000079	1·221881	1·000086	1·119244
−1·500175	1·400587	1·500104	1·330404
−2·000504	1·628145	2·000592	1·575440
−2·500929	1·882834	2·500329	1·840947
−3·001445	2·152953	3·000222	2·118186

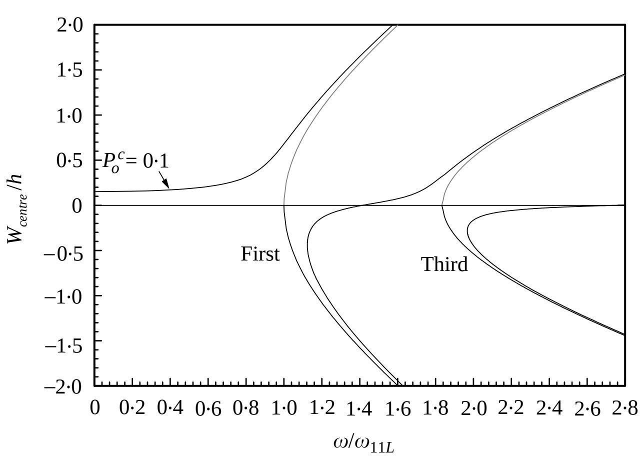

Figure 8. Forced vibrations of a fully clamped rectangular plate under a harmonic concentrated force at the centre ($L/l = 2$, $L/h = 240$).

the non-linear resonance curves at any desired range of amplitudes with a small computation time. Numerical results for non-linear frequency and non-linear displacements are presented and compared for various types of excitations and plate boundary conditions. Presented results agree well with results available in the literature. One can wonder if a single-mode analysis is sufficient to account for large-amplitude vibrations. The latter method permitted one to obtain amplitude–frequency response curves in the vicinity of the first resonance particularly when the mode interactions are very small and the excitation is uniformly distributed or concentrated at the centre. But, this estimation loses its accuracy when the effect of higher modes is consequent. Moreover, it did not give any information about the non-linear effects on the displacement and the stress fields. This additional information has been obtained by the present method with a small computation time.

The study of large-amplitude vibrations of structures involving geometrical non-linearities requires efficient non-linear procedures permitting one to obtain not only the non-linear frequency but also the non-linear response of the structure at any desired point and subjected to various types of excitations. This method very well fitted these objectives and several examples of application have been described to illustrate the power of this method. Other related topics such as internal resonance and non-linear damped vibrations of thin elastic structures may be investigated using this technique. It can be incorporated in any finite element code in order to study the non-linear vibration of more complex structures with various types of excitations.

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APPENDIX A

The path parameter a in the series (18), identified as the projection of the displacement increment $(\mathbf{U}(a) - \mathbf{U}_0)$ and the frequency increment $(\omega(a) - \omega_0)$ on the tangent vector $(\mathbf{U}_1, \omega_1)$ [42], can be written as

$$a = 1/s^2\{\langle \mathbf{U} - \mathbf{U}_0, \mathbf{U}_1\rangle + (\omega - \omega_0)\omega_1\}, \tag{A.1}$$

where $\langle \cdots \rangle$ is the Euclidean scalar product and s is a scaling parameter which corresponds to the length of the tangent vector (U_1, ω_1). Introducing series (18) into equation (A.1) and equating like powers of a one obtains the set of single equations

$$\begin{aligned} &\textit{order } 1: \quad \langle \mathbf{U}_1, \mathbf{U}_1\rangle + \omega_1\omega_1 = s_2,\\ &\textit{order } 2: \quad \langle \mathbf{U}_1, \mathbf{U}_2\rangle + \omega_1\omega_2 = 0,\\ &\textit{order } 3: \quad \langle \mathbf{U}_1, \mathbf{U}_3\rangle + \omega_1\omega_3 = 0,\\ &\qquad\vdots\\ &\textit{order } p: \quad \langle \mathbf{U}_1, \mathbf{U}_p\rangle + \omega_1\omega_p = 0. \end{aligned} \tag{A.2}$$

All vectors $\mathbf{U}_p$ and coefficients ω_p of the series (18) can be determined by successively solving the systems of equations (19) and (A.2) at each order p.

By recalling that the unknown vectors $\mathbf{U}_p$ are mixed (displacement–stress), and following the same manipulation presented in papers [2, 3, 40–42], are returns to a pure displacement formulation. After discretization by FE, are obtains the matrix problem:

$$[\mathbf{K}_t(\mathbf{U}_0)]\{\mathbf{U}_p\} = \omega_p\{\mathbf{F}\} + \{\mathbf{F}_p^{NL}\}; \langle \mathbf{U}_1, \mathbf{U}_p\rangle + \omega_1\omega_p = 0, \tag{A.3}$$

where $[\mathbf{K}_t(\mathbf{U}_0)]$ is the tangent stiffness matrix at the starting point $(\mathbf{U}_0, \omega_0)$, $\{\mathbf{F}\}$ is the column vector $\{[\mathbf{M}]\{\mathbf{U}_0\}\}$, and $\{\mathbf{F}_p^{NL}\}$ represents the remaining part of the right side of equations (19) [3, 41]. Problem (A.3) is solved in the following steps:

At order 1:
Step 1: Solve $[\mathbf{K}_t(\mathbf{U}_0)]\}(\mathbf{U}_1^L) = \{\mathbf{F}\}$;

Step 2: Compute $\omega_1 = \pm s/\sqrt{\langle \mathbf{U}_1^L, \mathbf{U}_1^L \rangle + 1}, \quad \mathbf{U}_1 = \omega_1\, \mathbf{U}_1^L$;
Step 3: Compute $\mathbf{N}(1) = [\mathbf{C}_m]\{\boldsymbol{\gamma}^L(\mathbf{U}_1) + 2\boldsymbol{\gamma}^{NL}(\mathbf{U}_0, \mathbf{U}_1)\}$;

At order p:
Step 1: Solve $[\mathbf{K}_t(\mathbf{U}_0)]\{\mathbf{U}_p^{NL}\} = \{\mathbf{F}_p^{NL}\}$;
Step 2: Compute $\omega_p = (\langle \mathbf{U}_1, \mathbf{U}_p^{NL} \rangle / s^2)\, \omega_1$, $\mathbf{U}_p = (\omega_p/\omega_1)\, \mathbf{U}_1 + \mathbf{U}_p^{NL}$;
Step 3: Compute $\mathbf{N}(p) = [\mathbf{C}_m][\boldsymbol{\gamma}^L(\mathbf{U}_p) + 2\boldsymbol{\gamma}^{NL}(\mathbf{U}_0, \mathbf{U}_p) + \sum_{r=1}^{p-1} \boldsymbol{\gamma}^{NL}(\mathbf{U}_r, \mathbf{U}_{p-r})]$.